

Fluid Dynamics

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Problem 1 (Weighing in a Vacuum). Estimate how many extra grams a scale would read if you weighed yourself in a vacuum.

Problem 2 (Water and Oil). A U-shaped tube with two straight vertical arms, each of cross-sectional area 3 cm^2 , contains some water. Into one of the openings, 0.08 L of oil with density 920 kg/m^3 is poured. What is the absolute difference in height between the two free liquid surfaces?

Problem 3 (Superman and the Straw). Superman tries to drink water from a lake using a straw longer than 2 km , but he fails. What is the maximum height above the lake level that the water can reach inside the straw?

Problem 4 (Hot-Air Balloon). An inexperienced explorer is traveling in a hot-air balloon with total mass (explorer included) 350 kg . Due to some poor decisions, the balloon starts descending with acceleration 1.5 m/s^2 . To clear a hill ahead, the explorer drops ballast until the balloon accelerates upward at 1 m/s^2 . What is the mass of the ballast?

Problem 5 (Force on a Dam). A dam of width L retains the water of a lake to height H . What is the magnitude of the total force exerted on the dam?

Problem 6 (Dry Dock). Consider the floating problem of a ship in a dry dock, simplified as follows: the dock is a right rectangular parallelepiped of width a and length b , and the ship is a right prism of the same length with an equilateral triangular base of side $a/2$ and uniform density δ , slightly less than that of water. Assume further that the top face of the prism is horizontal when the ship floats. Compute the minimum volume of water that must be added to the dock to make the ship float.

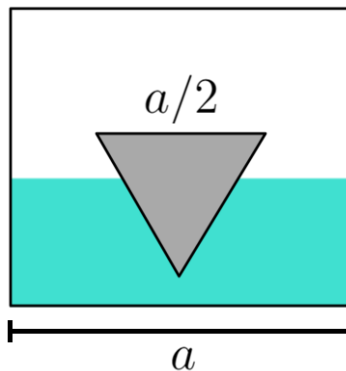


Figure 1: Cross-section of the dry dock.

Problem 7 (Air Bubble). An air bubble is released at the bottom of a closed cylinder, 10 m tall and filled with water. At the instant the bubble is injected, the pressure at the cylinder bottom is 1.1 atm . The bubble rises until it reaches the top of the cylinder. What is the pressure at the bottom after the bubble has risen?

Problem 8 (Helium Balloon). A balloon is inside a truck that suddenly accelerates forward. What happens to the balloon in the truck's reference frame?

Problem 9 (Faucet Jet). From a faucet of cross-section A_0 , water exits with speed $\vec{v}_0$ directed downward. Find the cross-section of the jet at a generic height z below the faucet, if the faucet mouth is at height h .

Problem 10 (Parabolic Jets). A cylindrical container of height $H = 50$ cm is placed on a pedestal of height $h = 50$ cm and filled with water to the brim. A small hole is made in the lateral surface near the bottom of the cylinder, and water starts to flow out. What is the maximum horizontal distance reached by the water jet?

Problem 11 (Emptying a Bucket). Compute the emptying time of a cylindrical bucket of height h_0 and base area A if a hole of cross-section $S \ll A$ is made in the bottom.

Problem 12 (Accelerating Cart). A cart containing water moves with uniform linear acceleration $\vec{a} = a \hat{x}$. Determine the free-surface shape of the water.

Problem 13 (Rotating Bucket). A cylindrical container of radius R , filled with volume V of a liquid, rotates about its axis with angular speed ω . Find the shape of the liquid free surface once a steady state is reached.

Problem 14 (Container Shape). Determine the shape a container with cylindrical symmetry must have so that the water level drops at a constant rate when a very small hole is opened at the bottom.

Problem 15 (Two Buckets). Consider two identical buckets filled with water, each with a hole in the bottom connected to a vertical tube (straw). The tubes have the same cross-section, but one is longer than the other. Neglecting viscosity losses, which bucket empties first?

Problem 16 (Magnus Effect). A baseball can be thrown with some angular speed about its axis to produce the so-called “curveball,” sketched in the figure. Estimate the lateral force F_l and the deflection d using dimensional analysis and the data provided below. Assume F_l is directly proportional to ω and that the dimensionless proportionality constant is of order unity.

$$\begin{aligned} m &= 0.145 \text{ kg}, & R &= 3.7 \text{ cm}, & v &= 36 \text{ m/s}, \\ \omega &= 227 \text{ rad/s}, & L &= 18 \text{ m}, & \rho_a &= 1.2 \text{ kg/m}^3. \end{aligned}$$

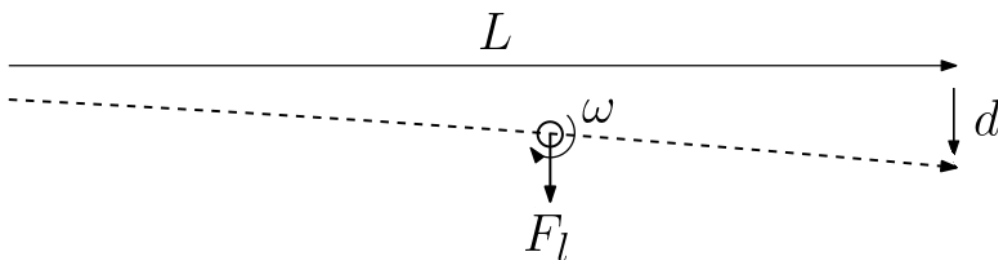


Figure 2: Top view of the baseball trajectory.

Problem 17 (Falling Sphere). A small iron sphere of volume V sinks at constant speed v in a container full of water.

1. What is the total momentum of the water?
2. If the container is resting on a table while the sphere descends, what is the force exerted by the table on the container?

Solutions

Solution 1 (Weighing in a Vacuum). To estimate the buoyant (Archimedes) force acting on our body, we just need its volume. A reasonable estimate is 70 liters, i.e. about $7 \times 10^{-2} \text{ m}^3$. For a more accurate value, one could submerge in a pool and measure the change in water height to infer the displaced volume. The buoyant force due to air is roughly $F = \rho_{\text{air}} g V \approx 0.8 \text{ N}$, which corresponds to just under 100 g on a scale.

Solution 2 (Water and Oil). Because oil is immiscible and less dense than water, it floats on top, forming a column of height $h_{\text{oil}} = V/S$, where V is the oil volume and S is the tube cross-section. The pressure at the free surface in both arms is atmospheric, p_0 , while the pressure at the base of the oil column follows Stevin's law:

$$p_{\text{oil}} = p_0 + \rho_{\text{oil}} g h_{\text{oil}},$$

where ρ_{oil} is the oil density. In hydrostatic equilibrium, the pressure at the same level in the other arm must match:

$$p_{\text{oil}} = p_{\text{atm}} + \rho_{\text{w}} g h_{\text{w}},$$

where h_{w} is the height of the water column above that level. Hence $h_{\text{w}} = h_{\text{oil}} \rho_{\text{oil}} / \rho_{\text{w}}$, and the height difference is

$$\Delta h = h_{\text{oil}} - h_{\text{w}} = \frac{V}{S} \left(1 - \frac{\rho_{\text{oil}}}{\rho_{\text{w}}} \right).$$

Solution 3 (Superman and the Straw). The pressure at the lower end of the straw (at the lake surface) must be atmospheric, p_0 , while at the top it can be, at best, zero relative to vacuum. Applying Stevin's law, the maximum water height in the straw is that which produces atmospheric pressure at the base:

$$h_{\text{max}} = \frac{p_0}{\rho g}.$$

Any greater height would make the water leave the lower end, reducing the column back to h_{max} .

Solution 4 (Hot-Air Balloon). Two forces act on the balloon: weight and buoyancy F_A . The buoyant force depends on the balloon volume, which changes by a negligible amount after dropping ballast. The initial acceleration is therefore

$$a_0 = g - \frac{F_A}{m},$$

where m is the total mass. After releasing ballast of mass m_Z , the acceleration becomes

$$a_1 = g - \frac{F_A}{m - m_Z}.$$

These two equations form a system for the unknowns F_A and m_Z . Solving gives

$$m_Z = m \frac{a_0 - a_1}{g - a_1}.$$

With our sign convention, $a_0 > 0$, $g > 0$, and $a_1 < 0$.

Solution 5 (Force on a Dam). Consider an infinitesimal horizontal strip of the dam at depth y , of area $dA = L dy$. The infinitesimal force on this strip is

$$dF = P(y) dA = \rho g y L dy.$$

Integrating from 0 to H gives the total force

$$F = \int_0^H \rho g y L dy = \frac{1}{2} \rho g L H^2.$$

Solution 6 (Dry Dock). Floating occurs when the buoyant force equals the ship's weight. Suppose that at some moment the water height in the dock is h and the ship still touches the bottom. The displaced water volume is

$$V(h) = \frac{h^2 b}{\sqrt{3}}.$$

Let ρ be the water density. The floating condition is

$$\rho g V(h) = M_{\text{ship}} g,$$

from which we find the height h at which the ship lifts off the bottom:

$$h_m = \frac{a}{4} \sqrt{\frac{3\delta}{\rho}}.$$

Hence the required water volume is

$$V_A = abh_m - V(h) = \sqrt{\frac{3\delta}{\rho}} \left(\frac{a^2 b}{4} \right) \left(1 - \frac{1}{4} \sqrt{\frac{\delta}{\rho}} \right).$$

Solution 7 (Air Bubble). During the bubble's rise, the water can be treated as incompressible and, to a very good approximation, its volume remains constant. Consequently, the bubble volume also remains constant (its temperature stays equal to the water's). If volume and temperature are unchanged, the bubble pressure stays constant at 1.1 atm. Once the bubble reaches the top, the water at that point is at 1.1 atm; therefore, by Stevin's law, the bottom pressure is 2.1 atm.

Solution 8 (Helium Balloon). The balloon is overall less dense than the surrounding air. In the truck's accelerating frame, the fictitious force on the balloon is smaller than that on an equal volume of displaced air, so pressure gradients push it forward—in the direction of the truck's acceleration—until it hits the front wall. Equivalently: in the accelerating frame there is an effective gravity along $\hat{x}$; by analogy with the vertical case, a lighter object moves opposite to that effective force. For the same reason, if the truck turned right, the balloon would drift to the right in the truck's frame.

Solution 9 (Faucet Jet). Since the density is constant,

$$v_z A(z) = v_0 A_0 \quad \implies \quad A(z) = \frac{v_0 A_0}{v_z} = \frac{A_0}{\sqrt{\frac{2g(h-z)}{v_0^2} + 1}},$$

where v_z is obtained from mechanical-energy conservation. Thus, the jet narrows as it falls because its speed increases¹.

Solution 10 (Parabolic Jets). By Torricelli's law, the exit speed v at the hole is

$$v = \sqrt{2gx},$$

where x is the height of the free surface above the hole. The maximal initial speed, which sets the range of the jet, is therefore

$$v_{\text{max}} = \sqrt{2Hg}.$$

¹As the cross-section decreases, the radial velocity component v_r is nonzero. Strictly speaking, it should be included in the energy balance. However, if v_0 is large enough, $v_r \ll v_z$, so its contribution can be neglected.

The water leaves horizontally and follows a parabolic trajectory:

$$\begin{cases} x = v_{\max}t, \\ y = h - \frac{1}{2}gt^2. \end{cases}$$

Setting $y = 0$ and solving for x gives

$$x = v_{\max}\sqrt{\frac{2h}{g}} = 2\sqrt{hH}.$$

Solution 11 (Emptying a Bucket). Given the small outlet area, the flow can be taken as quasi-steady. Let v be the speed just outside the hole; Bernoulli yields

$$v = \sqrt{2gh},$$

with h the liquid height. The free-surface speed relates to the outflow rate, which is minus the time derivative of the contained volume:

$$vS = -A \frac{dh}{dt}.$$

Substituting into Torricelli's law,

$$\frac{dh}{dt} = -\frac{S}{A}\sqrt{2gh},$$

whose solution is

$$h(t) = \left(\sqrt{h_0} - \frac{St}{2A}\sqrt{2g} \right)^2.$$

Thus, the emptying time is

$$t = \frac{A}{S}\sqrt{\frac{2h_0}{g}}.$$

Solution 12 (Accelerating Cart). In the accelerating frame we have

$$\phi = \frac{p}{\rho} + \frac{v^2}{2} + gz + ax = \text{constant}.$$

At equilibrium the fluid velocity is zero everywhere, so the equipotential (and isobaric) surfaces satisfy

$$z(x) = \text{constant} - \frac{a}{g}x,$$

i.e. planes inclined with respect to the cart's base.

Solution 13 (Rotating Bucket). There are two approaches. One is a free-body diagram of a small fluid element (left as an exercise). The second cleverly exploits Bernoulli invariance by working in a frame centered on the cylinder axis and rotating with the fluid at angular speed ω . In this frame the fluid is at rest, the flow is irrotational, and Bernoulli gives

$$B' = \frac{p(z, r)}{\rho} + \phi' + \frac{1}{2}v'^2(z, r) = \frac{p_0}{\rho} + \left(gz - \frac{1}{2}\omega^2 r^2 \right) + 0,$$

using that pressure is a scalar (frame-invariant), the velocity vanishes everywhere, and the potential is

$$\phi' = gz - \frac{1}{2}\omega^2 r^2,$$

the sum of gravitational and centrifugal potentials. (In this non-inertial frame, Coriolis gives no contribution because $v = 0$.) Hence,

$$z = \frac{B' - p_0}{\rho g} + \frac{\omega^2}{2g} r^2,$$

a paraboloid opening upward. The constant B' follows from volume conservation:

$$V = \int_0^R \left(\frac{B' - p_0}{\rho g} + \frac{\omega^2 r^2}{2g} \right) 2\pi r \, dr = \pi R^2 \left(\frac{B' - p_0}{\rho g} + \frac{\omega^2 R^2}{4g} \right).$$

Thus the free-surface shape is

$$z = \frac{V}{\pi R^2} + \frac{\omega^2}{4g} (2r^2 - R^2),$$

independent of the liquid density. Note that for very large ω , z can become negative at $r = 0$: the liquid then does not contact the rotation axis and piles up near the walls.

Solution 14 (Container Shape). Let A be the outlet area. For axially symmetric vessels, the free-surface area is $S = \pi r^2(h)$, where r is the radius of the circular section at height h . Conservation of flow rate gives

$$S \left| \frac{dh}{dt} \right| = A v,$$

where $v = \sqrt{2gh}$ by Bernoulli. We want the level to drop at a constant rate, so set $\frac{dh}{dt} = c$. Substituting,

$$c\pi r^2 = A\sqrt{2gh} \quad \Rightarrow \quad h(r) = k r^4,$$

with constant k . Revolving this quartic ($y = x^4$ type) around the y -axis gives the desired shape. Keep in mind Torricelli's law breaks down when h becomes comparable to the outlet size.

Solution 15 (Two Buckets). The exit speed at the lower end of the tube is

$$v = \sqrt{2gH},$$

where H is the head (pressure difference) between the free surface and the tube outlet. A longer tube places the outlet lower, increasing H ; thus v is larger and the bucket with the longer tube empties first.

Solution 16 (Magnus Effect). The force F_l is independent of the ball mass and travel distance, as it arises from interaction with the surrounding fluid. Using the given proportionality $F_l \propto \omega$,

$$F_l = C\rho_a^\alpha \omega v^\beta R^\gamma,$$

which dimensionally implies

$$\frac{[M][L]}{[T]^2} = \left(\frac{[M]}{[L]^3} \right)^\alpha \frac{1}{[T]} \left(\frac{[L]}{[T]} \right)^\beta [L]^\gamma,$$

yielding $\alpha = 1$, $\beta = 1$, $\gamma = 3$. With $C \approx 1$ as suggested,

$$F_l \approx \rho_a \omega v R^3 \approx 0.5 \text{ N}.$$

Assuming a small lateral deflection, take F_l perpendicular to the initial flight direction throughout. The trajectory is approximately parabolic, so

$$d \approx \frac{1}{2} a_l t^2, \quad a_l = \frac{F_l}{m}, \quad t = \frac{L}{v},$$

giving

$$d \approx \frac{F_l L^2}{2mv^2} = \frac{\rho_a \omega L^2 R^3}{2mv} \approx 0.44 \text{ m}.$$

Solution 17 (Falling Sphere). As the sphere moves, the system's mass distribution changes. After time t , the center of mass has shifted by

$$s = vt \frac{\rho_s V - \rho_w V}{M},$$

where M is the total mass, V the sphere volume, and ρ_s , ρ_w are the densities of the sphere and water, respectively. The idea is that as the sphere descends by vt , one may imagine an equal water sphere occupying its previous position. The total momentum of the system is then

$$p_{\text{tot}} = \frac{Ms}{t} = v\rho_s V - v\rho_w V,$$

the first term being the iron sphere's momentum and the second that of the (advected) water sphere. Since the system center of mass moves at constant speed, the net external force vanishes. Consequently, the force the table exerts on the container equals and opposes the system's total weight.