Quirky Problems from the Classical Mechanics and Thermodynamics Course 2020/21

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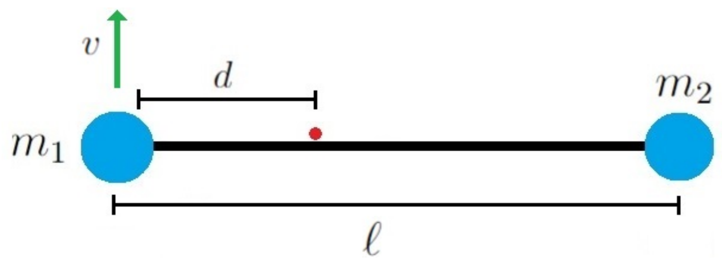
1 Kinematics and Dynamics

Problem 1.1 (Bicycle Wheel). Imagine taking a photo of a rolling bicycle wheel that does not slip. The wheel can be schematized as a circle connected to its center by a large number of spokes. Since, to take a photo, the camera lens is opened for a short but non-zero time δt, most of the image will appear blurred because the elements of the wheel move during the lens opening. However, there are points where the photo is perfectly in focus. Find them.

Problem 1.2 (Motion of Two Masses on a Table). Two point masses m_1 and m_2 are connected by an inextensible string of length ℓ and negligible mass. They are placed on a perfectly smooth horizontal table. The following figure shows the configuration from above (the dimensions of the two bodies are exaggerated).



The red dot in the figure represents a nail fixed to the table, initially at a distance d from the body on the left. At the initial moment, the string is perfectly taut and an impulse is given to the left body, imparting a velocity v perpendicular to the string.

1. Using the position of the nail as the origin, find the trajectories of the two bodies from the initial moment until the mass m_2 reaches/passes the nail. How much time elapses between these two moments?

2. Within the same time domain as the previous point, what is the minimum distance between the two bodies?

3. After the second body has passed the nail, what type of motion do the two bodies exhibit? Describe it.

Problem 1.3 (Atwood Machine with Massive String). An Atwood machine consists of a pulley with negligible dimensions and two blocks of masses m_1 and m_2, connected by a homogeneous cable of length $2l$ and mass M. Initially, the blocks are at the same height and then released. Find the function $y(t)$ that describes the temporal evolution of the system until the lighter body collides with the pulley.

Problem 1.4 (Non-Homogeneous Rough Plane). A plane is made of different materials, arranged in sectors as in Fig. 1. The N sectors all meet at the origin and have different angular openings θ_i such that $\sum_{i=1}^{N} \theta_i = 2\pi$. Each sector is characterized by the friction coefficient μ_i.

A homogeneous disk of radius R and mass M is set into rotation around its axis with angular velocity $\vec{\omega} = \omega\hat{z}$ and is placed on the plane so that its center coincides with the origin.

1. What is the acceleration vector of the disk immediately after it is placed on the ground?
2. Find its initial angular acceleration
3. What happens after the initial moment?

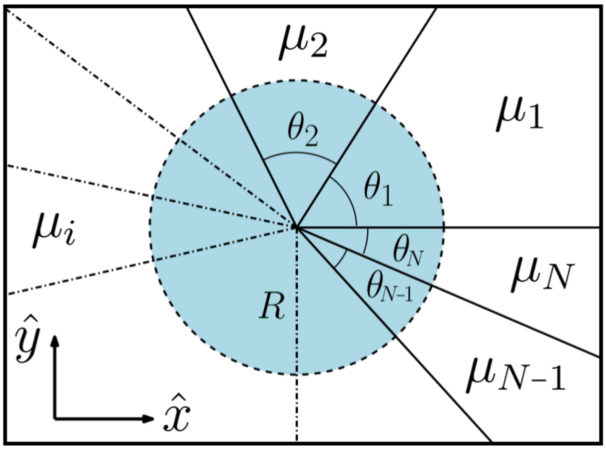


Figure 1: Disk placed on a rough plane.

2 Gravitation and Central Forces

Problem 2.1 (Circular Orbit). A particle of mass m is subjected to a central force that lets its orbit to be described by the polar equation $r(\theta) = 2R\cos\theta$. Find the orbital period of such orbit as a function of **only** m, R, and the force parameter.

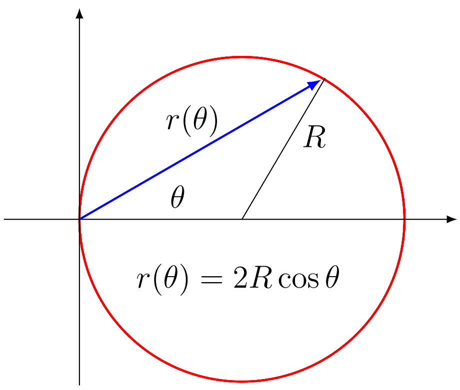


Figure 2: Circular orbit.

Problem 2.2 (Cardioid Orbit). A particle of mass m is subjected to a central force that lets its orbit to be described by the polar equation $r(\theta) = R(1 + \cos\theta)$. Find the orbital period as a function of **only** m, R, and the force parameter.

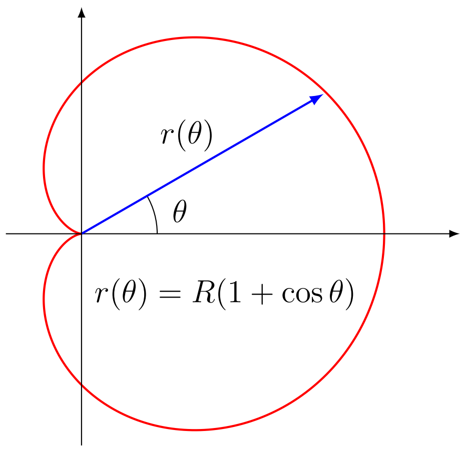


Figure 3: Cardioidal orbit.

Problem 2.3 (Ballistic Missile). A ballistic missile is launched from a point A on the Earth's surface with an initial velocity $\vec{v}_0$, whose magnitude is less than the escape velocity, at an angle α from the horizontal. After traveling on an elliptical trajectory, the missile crashes at a certain point B, whose position depends on the launch conditions. The situation is illustrated in Fig. 4.

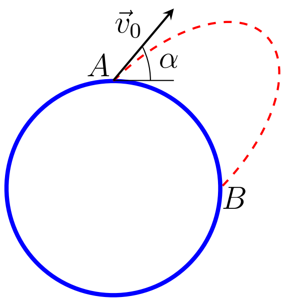


Figure 4: Example of projectile motion within Earth's gravitational field.

1. Find the parameters of the elliptical orbit as a function of v_0 and α.
2. Determine the flight time and the distance between the launch and impact points on the Earth's surface. How far is the highest point from the Earth's surface?
3. What should the initial velocity v_0 be for a missile launched at a given angle α so that it impacts at the antipode, as shown in Fig. 5?

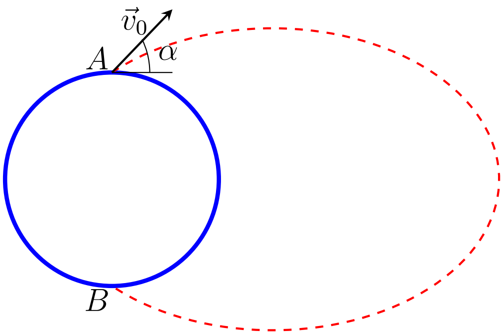


Figure 5: Projectile motion toward the antipode.

Problem 2.4 (Laplace–Runge–Lenz Vector). Find the equation of the orbits starting from the Laplace–Runge–Lenz vector:

$$\vec{A} = \vec{p} \times \vec{L} - m\alpha\hat{r}.$$

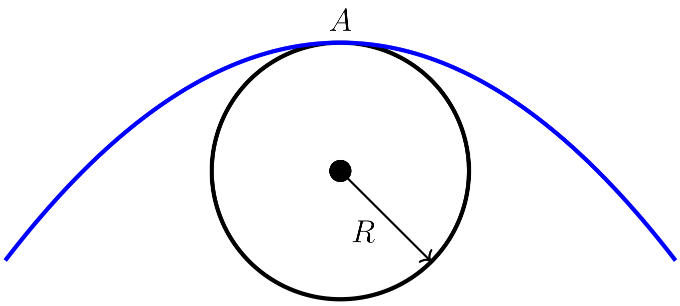
Problem 2.5 (Asteroid in the Solar System). An asteroid of mass m moves along a parabolic trajectory around the Sun in the same plane as Earth's orbit, which is assumed to be circular with radius R. Let p be the minimum distance between the asteroid and the Sun. Neglect the gravitational attraction between the asteroid and the Earth.

1. What is the angular momentum of the asteroid?
2. If $p < R$, what is the time T the comet spends inside Earth's orbit?
3. What is the maximum possible value of T?

If p is smaller than the Sun's radius, the asteroid will fall into it, increasing the mass of the Sun by a fraction $\alpha \equiv m/M_S$.

4. How do the parameters of Earth's orbit change as α varies?

If instead $p = R$, the parabola is tangent to the circle at a point, as shown in the following figure.



Suppose that the asteroid undergoes an inelastic collision with the Earth at point A, embedding itself into it.

5. How do the parameters of Earth's orbit change?
6. How much heat is released as a result of the collision?

Problem 2.6 (L4 and L5). A planet of mass m orbits a star of mass $M \gg m$ in a circular orbit. Find the positions of the Lagrangian points L4 and L5.

3 Rigid Body

Problem 3.1 (Rigid Body in Orbit). A satellite orbits the Earth in a circular trajectory of radius R. Calculate the torque with respect to the center of mass, given the inertia tensor $\overline{I}$ of the satellite. Initially, the satellite's three principal axes are oriented such that one is radial, one is tangential, and the last one is perpendicular to the orbital plane. Show that the orientation of the axes remains fixed in the frame rotating with the satellite. Compute the period of small oscillations in the orbital plane.

Problem 3.2 (**Asymmetric Top**). A rigid body with principal moments of inertia I_1, I_2, I_3 (with $I_1 < I_2 < I_3$) moves freely without external forces, with angular momentum $\vec{J} \equiv Jz$. The principal axes of the rigid body are denoted as $\hat{n}_1$, $\hat{n}_2$, $\hat{n}_3$, while the fixed inertial reference frame axes are $\hat{x}$, $\hat{y}$, $\hat{z}$. At time $t = 0$, the body is positioned such that the axis $\hat{n}_3$ forms an angle $\theta_0 \ll 1$ with the $\hat{z}$-axis, and the component of the angular velocity perpendicular to $\hat{n}_3$, ω_T^0, is much smaller than the parallel component, ω_3^0.

1. Determine the time dependence of the angular momentum vector $\vec{J}$ along the three principal axes of the top, choosing the time origin appropriately.
2. Write the time evolution equations for Euler angles θ, ψ, and ϕ.
3. Determine how the unit vector $\hat{n}_3$ evolves over time in the fixed inertial frame, decomposing it into its coordinates (n_{3x}, n_{3y}, n_{3z}).
4. Consider now an asymmetric top as described above, where the principal moments of inertia are defined as $I_1 = A - \Delta$, $I_2 = A + \Delta$, $I_3 = 2A$ (with $0 < \Delta < A$), and with general initial conditions. Show that the component ω_T of the angular velocity perpendicular to the axis $\hat{n}_3$ remains constant over time and derive the equation of motion for ω_3 to exploit this condition.

4 Calculus of Variations and Lagrangians

Problem 4.1 (**Motion of a Charged Particle**). Using the Lagrangian formalism, study the motion of a particle with charge q subject to a central force $\vec{F} = f(r)\,\hat{r}$ and a uniform magnetic field. Identify a cyclic coordinate and the corresponding conserved conjugate momentum. Provide a physical interpretation of this conservation law.

Problem 4.2 (**Chain of Springs**). A system consists of $N + 1$ springs of natural length L connecting N point masses of mass m, forming a linear chain with fixed endpoints separated by a distance $(N + 1)L$. Write the Lagrangian of the system in terms of N generalized coordinates describing the displacements from the equilibrium positions and determine the spectrum of the N normal mode frequencies of the system.

Problem 4.3 (**Maximal Gravitational Field**). We have a piece of modeling clay with mass m and uniform density ρ, and we want to shape it to obtain the maximum possible gravitational field at a given point in space. What shape should it have? What is the maximum gravitational field that can be achieved?

Problem 4.4 (**Maximal Gravitational Field in 2D**). Repeat the calculations of the previous problem in the case of a 2D universe, where the gravitational field has a different functional form.

Problem 4.5 (**Snell's Law for Spherically Symmetric Media**). Determine how Snell's law is written in the case of spherically symmetric media, characterized by a refractive index $n = n(r)$.

Problem 4.6 (**Light Ray Trajectories**). In the desert, the refractive index depends on altitude according to the law

$$n(z) = n_0\sqrt{1 + z/h},$$

where $n_0 > 1$ and h is a characteristic length. Find the function $z(x)$ describing the trajectory of a light ray that starts from point $A = (0, h)$ and reaches an observer at $B = (h, h)$.

Problem 4.7 (**Superior and Inferior Mirages**). Determine the conditions on the refractive index of air $n = n(z)$ required for the occurrence of superior and inferior mirages.

5 Thermodynamics

Problem 5.1 (Leaking Container). Inside a container of volume V, whose walls are perfect thermal insulators, there is an ideal monoatomic gas with molecular mass m, density ρ_0, and temperature T_0. The container is placed in a perfect vacuum. A hole of diameter d, smaller than the mean free path of the gas molecules, is opened on one of the walls of the container.

- Assuming that the temperature of the gas inside the container remains constant over time, what is the law that describes the variation of the gas density $\rho(t)$?
- The assumption made in the previous point is incorrect. Actually, the temperature changes because faster molecules escape before slower ones. What is the correct expression for $\rho(t)$?

Problem 5.2 (Two Compartments). A thermally insulated container is divided into two compartments, each containing the same number of monoatomic gas molecules. Initially, each compartment has the same temperature T_0, the same pressure p_0, and the same volume. As shown in Fig. 6, two pistons are attached to walls 1 and 3 (which do not conduct heat), while wall 2 is fixed and is a perfect heat conductor. The initial distance between walls 1 and 2 is L, and the same holds for walls 2 and 3. At a certain point, the pistons are activated, moving both walls 1 and 3 by $L/2$ to the right. The process is slow enough to allow the two compartments to reach thermal equilibrium. What is the final temperature of the gas after the process is completed?

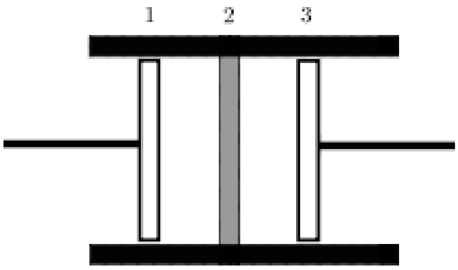


Figure 6: Compartments and pistons.

Problem 5.3 (Equation of State). Determine the equation of state $f(V, P, T) = 0$ of a classical gas of N particles, using *only* the following information: at constant temperature, its internal energy does not depend on volume and its enthalpy does not depend on pressure.